

On the energy landscape of 3D spin Hamiltonians with topological order

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We explore feasibility of a quantum self-correcting memory based on 3D spin Hamiltonians with topological quantum order in which thermal diffusion of topological defects is suppressed by macroscopic energy barriers. To this end we characterize the energy landscape of stabilizer code Hamiltonians with local bounded-strength interactions which have a topologically ordered ground state but do not have string-like logical operators. We prove that any sequence of local errors mapping a ground state of such Hamiltonian to an orthogonal ground state must cross an energy barrier growing at least as a logarithm of the lattice size. Our bound on the energy barrier is shown to be tight up to a constant factor for one particular 3D spin Hamiltonian.

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Topologically ordered phases of matter display a variety of fascinating properties having no counterpart in the classical physics. Most notable ones are topological invariants such as the the Hall conductance [1], ground state degeneracy [2], and topological entanglement entropy [3, 4] which are insensitive to generic local perturbations [5–7]. The intrinsic stability against perturbations motivated interest to topological phases as a storage medium for a reliable quantum memory [6, 8, 9] and as a platform for building a topological quantum computer [6, 10].

A big open question in the theory of topological quantum order (TQO) concerns feasibility of a non-volatile, or, *self-correcting*, quantum memory [6, 11]. Such a memory would permit reliable long-term storage of quantum information in a presence of sufficiently weak thermal noise without need for active stabilization and error correction during the storage period. The main challenge in designing Hamiltonians with self-correcting properties is to combine TQO with an energy landscape that could prevent errors caused by thermal fluctuations from accumulating. This could guarantee that the error density remains sufficiently small during the entire storage period and the encoded information can be safely extracted from the memory by performing an active error correction at the read-out phase.

In spite of being intrinsically stable against perturbations at the zero-temperature, TQO models display extreme fragility against thermal fluctuations [12] suggesting impossibility of quantum self-correction. A thermal stability analysis involving finite-temperature extensions of the topological entanglement entropy has been undertaken for the 2D and 3D toric code models by Castelnovo and Chamon [13], and by Iblisdir et al [14]. These models were shown to undergo a transition from a topologically ordered phase at $T = 0$ to a different phase with either partial or no topological order at any positive temperature [13–15].

The first rigorous analysis of self-correcting properties

for the toric code models was performed by Alicki et al [16, 17]. It showed that the 4D toric code Hamiltonian has self-correcting properties for sufficiently small temperature, while 2D and 3D toric codes are not self-correcting at any finite temperature. The ideas of [16, 17] were developed further by Kay [18], Chesi et al [19, 20] and Pastawski et al [21].

The main feature of the 4D toric code model responsible for self-correction is a *macroscopic energy barrier* that must be crossed by any sequence of local errors whose combined action on encoded states cannot be corrected at the final read-out phase [17]. The height of this barrier grows linearly with the lattice size due to a finite string-tension characterizing boundaries of membranes associated with errors. It is analogous to the energy barrier separating ground states with positive and negative magnetization in the ferromagnetic 2D Ising model. Unfortunately, this behavior cannot be reproduced in any known 2D or 3D model due to a presence of point-like excitations carrying a non-trivial topological charge, or, *point-like defects* [22]. These defects are analogous to domain-walls in the 1D Ising model — a single isolated defect has only a constant energy cost, but its creation requires a highly non-local operation affecting a macroscopic number of qubits (spins). Whether or not the presence of point-like defects rules out self-correcting properties may depend on how fast these defects can diffuse across the system. For example, Hamma et al [23] used a coupling with a bosonic field to create an effective long-range attractive interaction between defects whereby suppressing the diffusion. A different possibility is realized in the 3D Chamon’s model [24, 25]. This model offers a topological protection against diffusion of some types of defects (but not all of them). These defects, called monopoles in [25], can be created at corners of rectangular shaped membranes. A hopping of a single isolated monopole between adjacent lattice sites is a highly non-local operation affecting a macroscopic number of qubits, see [25] for details.

In the present paper we propose yet another possibility to suppress the diffusion of defects that can be realized in certain 3D spin Hamiltonians with strictly local bounded-strength interactions. These Hamiltonians, associated with stabilizer error correcting codes [26], have a peculiar property that isolated defects cannot move further than a certain constant distance away without creating other defects. For brevity, we shall refer to this property (stated more formally below) as a *no-strings rule* because it is closely connected to the absence of logical string-like operators capable of moving the defects. Let us point out that the first example of a 3D spin Hamiltonian with TQO obeying the no-strings rule has been found only quite recently by one of us [27]. We prove that any sequence of local errors creating an isolated defect from the vacuum with no other defects within distance R must cross an energy barrier at least $c \log R$ for some constant c . Furthermore, the length of such error sequence must grow at least as R^γ for some constant $\gamma > 1$. The same bound applies to creation of any isolated cluster of defects with a non-trivial total topological charge.

It shows that although defects do not interact directly, their diffusive motion is suppressed by logarithmic energy barriers preventing the defects from spreading (the concept of a diffusive motion must be used with care in our case because individual defects can only move a constant distance away).

We also prove a logarithmic lower bound on the energy barrier for implementing any logical operator. More precisely, we prove that any sequence of local errors mapping a ground state to an orthogonal ground state must cross an energy barrier at least $c \log L$, where L is the lattice size and c is some constant. For the Hamiltonian discovered in [27] this bound is tight up to a constant factor. Although the scaling of the energy barrier is not as favorable as the one in the 4D toric code, we point out that the energy barrier does not grow with the lattice size at all for all previously studied TQO Hamiltonians in the 2D and 3D geometry. A naive estimate of the storage time τ for a memory with an energy barrier B operating at a temperature T can be made using the Arrhenius law, namely, $\tau \sim e^{B/T}$. Since in our case $B = c \log L$ for some constant c , we arrive at $\tau \sim L^{c/T}$. Although this ‘derivation’ gives only polynomial scaling of τ , the degree of the polynomial can be made arbitrarily large by choosing sufficiently small temperature.

It is worth pointing out that 2D Hamiltonians with TQO always have string-like logical operators and thus 3D is the smallest spatial dimension for constructing Hamiltonians obeying the no-strings rule. Indeed, it was shown by Terhal and one of us [28] that for any 2D local stabilizer-type Hamiltonians the energy barrier for implementing at least one logical operator is constant. It should also be noted that 3D translation-invariant stabilizer Hamiltonians with TQO can obey the no-strings rule only if the ground state degeneracy is not invariant

under changing lattice dimensions [29, 30].

An obvious difficulty in proving a lower bound on the energy barrier is that there are too many ways to choose an error path (i.e., a sequence of local errors) implementing a fixed Pauli operator. In fact, we do not impose any restrictions on the length of the error path except that it is finite. Our bound should be applicable to any such error path. An additional difficulty is that a Pauli operator that an error path needs to implement is only defined modulo stabilizers. We resolve these difficulties using a novel technique which can be regarded as a renormalization group in the space of error paths, see Fig. 1 for a brief summary of the technique. Let us now state our main results more formally.

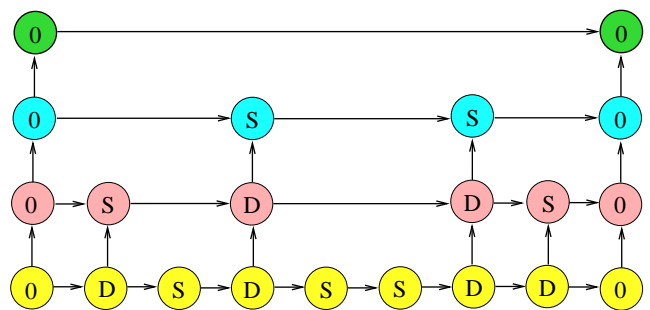


FIG. 1. Renormalization group technique used to prove a logarithmic lower bound on the energy barrier. Horizontal axis represents time. Vertical axis represents RG level $p = 0, 1, \dots, p_{\max}$. A sequence of level-0 errors (single-qubit Pauli operators) implementing a logical operator \bar{P} defines a level-0 syndrome history (yellow circles) that consists of sparse (S) and dense (D) syndromes. The history begins and ends with the vacuum (0). For any level $p \geq 1$ we define a level- p syndrome history by retaining only dense syndromes at the lower level. A syndrome is called dense at level p if it cannot be partitioned into clusters of size $\leq (10\alpha)^p$ separated by distance $\geq (10\alpha)^{p+1}$, where α is a constant coefficient from the no-strings rule. Each level- p error (horizontal arrows) connecting syndromes S', S'' is equivalent to the product of all level- $(p-1)$ errors between S', S'' modulo a stabilizer. We prove that these stabilizers can be chosen such that level- p errors act on $2^{O(p)}$ qubits. Since at the highest level $p = p_{\max}$ a single level- p error is a logical operator, one must have $p_{\max} = \Omega(\log L)$. We prove that level- p dense syndromes contain $\Omega(p)$ defects which implies that at least one syndrome at level $p = p_{\max} - 2$ consists of $\Omega(\log L)$ defects.

Stabilizer code Hamiltonians. Quantum spin Hamiltonians based on stabilizer codes provide convenient mathematical models of TQO that became an active research topic over last years. We consider a regular D -dimensional cubic lattice Λ with periodic boundary conditions and linear size L , that is, $\Lambda = \mathbb{Z}_L^D$. Each site $u \in \Lambda$ is populated by a finite number of qubits. A large class of TQO models can be described by the so-called

stabilizer Hamiltonians defined as

$$H = - \sum_{a=1}^M G_a, \quad (1)$$

where each term G_a is a multi-qubit Pauli operator (a tensor product of I, X, Y, Z with an overall ± 1 sign) and different terms commute with each other. Hence

$$G_a G_b = G_b G_a \quad \text{and} \quad G_a^2 = I. \quad (2)$$

The abelian group \mathcal{G} generated by G_1, \dots, G_M is called a *stabilizer group* of the code. Elements of \mathcal{G} are called *stabilizers*. Physically realistic Hamiltonians may only involve interactions between small subsets of qubits located close to each other. We shall assume that each generator G_a acts non-trivially (by X, Y or Z) only on a set of qubits located at vertices of an elementary cube. It is allowed to have more than one generator per cube. Any short-range stabilizer Hamiltonian can be written in this form by performing a coarse-graining of the lattice. The Hamiltonian may or may not be translation-invariant.

We shall assume that H is frustration-free [31], that is, ground states ψ_0 of H obey $G_a \psi_0 = \psi_0$ for all a , or, equivalently, the stabilizer group \mathcal{G} does not contain $-I$. Consider any multi-qubit Pauli operator E . A state $\psi = E \psi_0$ is an excited eigenstate of H . Obviously, $G_a \psi = \pm \psi$ where the sign depends on whether G_a commutes (plus) or anticommutes (minus) with E . Any flipped generator ($G_a \psi = -\psi$) will be referred to as a *defect*. Let us emphasize that a configuration of defects in ψ is the same for all ground states ψ_0 . An eigenstate with m defects has energy $2m$ above the ground state. For brevity, we shall use the term *vacuum* for a ground state of H whenever its choice is not important. A Pauli operator E whose action on the vacuum creates no defects is either a stabilizer ($E \in \mathcal{G}$), or a *logical operator* ($E \notin \mathcal{G}$, but E commutes with \mathcal{G}). In the former case any ground state of H is invariant under E . In the latter case E maps some ground state of H to an orthogonal ground state.

Topological order. A Hamiltonian is said to have topological order if it has a degenerate ground state and different ground states are locally indistinguishable. We shall need a slightly stronger version of this condition that involves properties of both ground and excited states. These properties depend on a length scale L_{tqo} that must be bounded as $L_{tqo} \geq L^\beta$ for some constant $\beta > 0$. (For stabilizer codes, L_{tqo} corresponds to the code distance.) Most of stabilizer code Hamiltonians with topological order satisfy our conditions with $L_{tqo} \sim L$. Our first TQO condition concerns ground states:

If a Pauli operator E creates no defects when applied to the vacuum and its support can be enclosed by a cube of linear size L_{tqo} then E is a stabilizer, $E \in \mathcal{G}$.

Our second TQO condition concerns excited states. A cluster of defects S will be called *neutral* if it can be created from the vacuum by a Pauli operator E whose support can be enclosed by a cube of linear size L_{tqo} without creating any other defects. Otherwise we say that S is a *charged* cluster. Given a region $A \subseteq \Lambda$ we shall use a notation $\mathcal{B}_r(A)$ for the r -neighborhood of A , that is, a set of all points that have distance at most r from A . Here and below we use l_∞ -distance on \mathbb{Z}_L^D . We shall need the following condition saying that neutral clusters of defects can be created from the vacuum locally [32].

Let S be a neutral cluster of defects and $C_{min}(S)$ be the smallest cube that encloses S . Then S can be created from the vacuum by a Pauli operator supported on $\mathcal{B}_1(C_{min}(S))$.

No-strings rule. We can now state the property of having no logical string-like operators. Informally, it says that applying an operator with a ‘string-like’ support to the vacuum cannot create charged defects at the end-points of the string, assuming that the string is sufficiently long. Let us now define this property rigorously. Let E be any Pauli operator whose support can be enclosed by a cube of linear size L_{tqo} and S be a cluster of defects obtained by applying E to the vacuum. Let A_1, A_2 be any pair of disjoint cubes of the same linear size ρ . We shall say that E is a *logical string segment* with *anchor regions* A_1, A_2 iff S is contained in the union $A_1 \cup A_2$. Equivalently, E must commute with all generators G_a located outside $A_1 \cup A_2$. We will say that a logical string segment E has *aspect ratio* α iff the distance between A_1 and A_2 is at least $\alpha \rho$. We shall only consider string segments with sufficiently large aspect ratio, say, $\alpha \geq 1$. A logical string segment E is called *trivial* iff the cluster of defects contained inside any anchor region is neutral.

A code has no logical string-like operators iff there exists a constant α such that all logical string segments with aspect ratio greater than α are trivial.

In the rest of the paper we shall abbreviate the condition of having no logical string-like operators as a *no-strings rule*. We note that a 3D stabilizer code (Code 1) discovered in [27] obeys our topological order conditions with $L_{tqo} \sim L$ and obeys the no-strings rule with $\alpha = 15$.

Energy barrier. Let us consider a process of building a logical operator \bar{P} from local errors. It can be described by an *error path* — a finite sequence of local Pauli errors E_1, \dots, E_T such that $\bar{P} = E_T \cdots E_2 E_1$. For simplicity we shall assume that each local error E_t is a single-qubit Pauli operator X, Y , or Z . Applying this sequence of errors to a ground state ψ_0 generates a sequence of states $\{\psi(t)\}_{t=0, \dots, T}$, where $\psi(0) = \psi_0$ and $\psi(T) = \bar{P} \psi_0$ are ground states of H , while the intermediate states $\psi(t) = E_t \cdots E_1 \psi_0$ are typically excited. We say that a logical operator \bar{P} has energy barrier ω iff for any error path

implementing \overline{P} at least one of the intermediate states $\psi(t)$ has more than ω defects. Note that we do not impose any restrictions on the length of the path T (as long as it is finite). In particular, one and the same error may be repeated in the error path several times at different time steps. We shall also consider an energy barrier for creating a cluster of defects S from the vacuum. We will say that S has energy barrier ω iff for any Pauli operator E that creates S from the vacuum and for any error path implementing E at least one of the intermediate states has more than ω defects.

Our main results are the following theorems. Both theorems apply to any stabilizer Hamiltonian Eq. (1) on a D -dimensional lattice that obeys the topological order condition and the no-strings rule.

Theorem 1. *The energy barrier for any logical operator is at least $c \log L$, where L is the lattice size, and c is a constant coefficient.*

Theorem 2. *Let S be a neutral cluster of defects containing a charged cluster $S' \subseteq S$ of diameter r such that there are no other defects within distance R from S' . If $r + R < L_{tqo}$, then the energy barrier for creating S from the vacuum is at least $c \log R$, where c is a constant coefficient.*

The constant c depends only on the spatial dimension D , the constant α in the no-strings rule, and the constant β in the bound $L_{tqo} \geq L^\beta$. Since the proof of both theorems uses the same technique, below we shall focus on proving Theorem 1. Proof of Theorem 2 will require only minor modifications that are explained later on.

Proof of Theorem 1. A configuration of defects created by applying a Pauli operator E to the vacuum will be called a *syndrome caused by E* . The process of building up a logical operator \overline{P} by a sequence of local errors E_1, \dots, E_T can be described by a *syndrome history* $\{S(t)\}_{t=0, \dots, T}$. Here $S(t)$ is the syndrome caused by the product $E_t \cdots E_1$, that is, the partial implementation of \overline{P} up to a step t . The syndrome history starts and ends with the vacuum, i.e., $S(0) = S(T) = \emptyset$. Without loss of generality all intermediate syndromes $S(t)$ are non-empty. For any integer $p \geq 0$ define a level- p unit of length as [33]

$$\xi(p) = (10\alpha)^p, \quad p = 0, 1, \dots$$

Let $S(t)$ be any non-empty syndrome. Recall that each defect in $S(t)$ can be associated with some elementary cube of the lattice.

Definition 1. *A syndrome $S(t)$ is called sparse at level p iff the set of elementary cubes occupied by $S(t)$ can be partitioned into a disjoint union of clusters such that each cluster has diameter at most $\xi(p)$ and any pair of distinct clusters combined together has diameter larger than $\xi(p+1)$. Otherwise $S(t)$ is called dense at level p .*

For example, suppose all defects in $S(t)$ occupy the same elementary cube. Since an elementary cube has diameter 1, such a syndrome $S(t)$ is sparse at any level $p \geq 0$. If $S(t)$ occupies a pair of adjacent cubes, $S(t)$ is sparse at any level $p \geq 1$, however it is dense at level $p = 0$. We note that the partition of $S(t)$ into clusters required for level- p sparsity is unique whenever it exists.

Lemma 1. *Suppose a non-empty syndrome $S(t)$ is dense at all levels $q = 0, \dots, p$. Then $S(t)$ contains at least $p+2$ defects.*

Proof. Let $C_1^{(0)}, \dots, C_g^{(0)}$ be elementary cubes occupied by $S(t)$. Obviously, $S(t)$ contains at least g defects. Since $S(t)$ is non-empty and dense at level 0, we have $g \geq 2$ and there exists a pair of cubes $C_a^{(0)}, C_b^{(0)}$ such that the union $C_a^{(0)} \cup C_b^{(0)}$ has diameter at most $\xi(1)$. Combining the pair $C_a^{(0)}, C_b^{(0)}$ into a single cluster we obtain a partition $S(t) = C_1^{(1)} \cup \dots \cup C_{g-1}^{(1)}$ where each cluster $C_a^{(1)}$ has diameter at most $\xi(1)$. Suppose $S(t)$ is dense at level 1. Then $g \geq 3$ and there exists a pair of clusters $C_a^{(1)}, C_b^{(1)}$ such that the union $C_a^{(1)} \cup C_b^{(1)}$ has diameter at most $\xi(2)$. Combining the pair $C_a^{(1)}, C_b^{(1)}$ into a single cluster and proceeding in the same way we arrive at $g \geq p+2$. \square

Let us define a level- p syndrome history as a subsequence of the original syndrome history $\{S(t)\}_{t=0, \dots, T}$ that includes only those syndromes $S(t)$ that are dense at all levels $q = 0, \dots, p-1$, see Fig. 1. The level-0 syndrome history includes all syndromes $S(t)$. The syndrome history starts and ends with the vacuum (empty syndrome) at any level p . Let $S(t')$ and $S(t'')$ be a consecutive pair of level- p syndromes. We define a level- p error E connecting $S(t')$ and $S(t'')$ as the product of all single-qubit errors E_j that occurred between $S(t')$ and $S(t'')$. Level- p errors are represented by horizontal arrows on Fig. 1. We would like to show that E can be regarded as an approximately local error on a coarse-grained lattice characterized by the unit of length $\xi(p)$. The problem however is that we do not have any bound on the number of single-qubit errors E_j in the interval between $S(t')$ and $S(t'')$. In the worst case, E could act non-trivially on every qubit in the system. The following lemma shows that level- p errors can be ‘localized’ by multiplying them with stabilizers. Let m be the maximum number of defects in the syndrome history, such that any $S(t)$ contains at most m defects.

Lemma 2. *Let $S' \equiv S(t')$ and $S'' \equiv S(t'')$ be a consecutive pair of syndromes in the level- p syndrome history. Let E be the product of all errors E_j that occurred between S' and S'' . If $4m(2 + \xi(p)) < L_{tqo}$, then there exists an error \tilde{E} supported on $\mathcal{B}_{\xi(p)}(S' \cup S'')$ such that $E\tilde{E}$ is a stabilizer.*

The proof of the lemma, presented in Appendix A, uses induction in the level p . The proof relies crucially on the no-strings rule. The latter asserts that an isolated charged defect belonging to some sparse syndrome cannot be moved further than distance α away by a sequence of local errors. Since the no-strings rule is scale invariant, it can also be applied to a coarse-grained lattice to show that isolated charged clusters belonging to some sparse level- p syndrome cannot be moved further than distance $\alpha\xi(p)$ away, see Appendix A for details.

Let p_{max} be the highest RG level, that is, the smallest integer p such that a single level- p error E maps the vacuum to itself, see Fig. 1. We claim that $p_{max} = \Omega(\log L)$. Indeed, suppose that $4m(\xi(p_{max}) + 2) < L_{tqo}$. Then we can apply Lemma 2 to the level- p_{max} syndrome history with $S' = S'' = \emptyset$ (vacuum). Lemma 2 would imply $\tilde{E} = I$, that is, E must be a stabilizer. On the other hand, E is equivalent to a logical operator modulo stabilizers. Hence we obtain a contradiction unless $4m(\xi(p_{max}) + 2) \geq L_{tqo}$. We can assume that the maximum number of defects is $m \ll \log L$ (if not, there is nothing to prove). Since L_{tqo} grows as a power of L , we conclude that $p_{max} = \Omega(\log L)$. The syndrome history must contain at least one syndrome $S(t)$ which is dense at all levels $q = 0, \dots, p_{max} - 2$ since otherwise p_{max} could not be the highest RG level. Lemma 1 then implies that such syndrome $S(t)$ contains $\Omega(\log L)$ defects proving Theorem 1. \square

We shall prove Theorem 2 using exactly the same approach as above. Let S be a neutral cluster of defects and E be a Pauli operator creating S from the vacuum, with $S' \subset S$ of diameter r being charged. Consider a hierarchy of syndrome histories similar to the one shown on Fig. 1, where we now maintain the initial syndrome \emptyset and the final syndrome S for all levels. Let p_{max} be the highest RG level. Then a single level- p_{max} error E creates S from the vacuum. Suppose $4m(\xi(p_{max}) + 2) < L_{tqo}$, where m is the maximum number of defects in the syndrome history. Lemma 2 implies that E is the equivalent modulo stabilizers to \tilde{E} supported on $\mathcal{B}_{\xi(p_{max})}(S)$. If $\xi(p_{max}) < R/4$, then \tilde{E} must act on two separated regions, one near S' and another far from S' . This means that S' alone can be created by a Pauli operator whose support is enclosed by a cube of linear size $r + R/2$. Since $r + R < L_{tqo}$, it is contradictory to the assumption that S' is charged. Therefore, $\xi(p_{max}) \geq R/4$, or $p_{max} = \Omega(\log R)$. In case where $4m(\xi(p_{max}) + 2) \geq L_{tqo} > R$, we also have $p_{max} = \Omega(\log R)$ provided $m \ll \log R$. (There is nothing to prove if $m \sim \log R$.) Since there must be a syndrome that is dense for all level- p where $p = 0, 1, \dots, p_{max} - 2$, Theorem 2 follows from Lemma 1.

Optimality. Our lower bound on the energy barrier is optimal up to a constant factor. As we noted earlier, Code 1 in [27] exhibits TQO with $L_{tqo} \sim L$ and obeys the no-strings rule with a constant $\alpha = 15$. This code can be

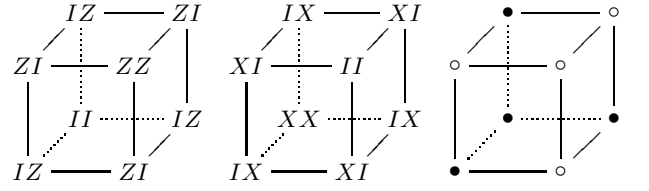


FIG. 2. Stabilizer generator for Code 1 in [27]. This translation-invariant model exhibits TQO, and obeys no-strings rule with $\alpha = 15$. The diagram on the right is a cube in the dual lattice. The filled dots indicate the defects created by XI at the center of the cube, which we call level-0 pyramid.

described by a stabilizer Hamiltonian Eq. (1) with two qubits per site and two stabilizer generators per cube as shown on Fig. 2. We shall use a notation PQ_u for a two-qubit Pauli operator $P \otimes Q$ applied to the pair of qubits located at site u . A single-qubit error XI_u creates a cluster of 4 defects (violated generators of Z -type) located at a set of cubes $S_c^{(0)} = \{c, c+\hat{x}, c+\hat{y}, c+\hat{z}\}$, where the center of c is obtained from u by a translation $-(\hat{x}+\hat{y}+\hat{z})/2$, see Fig. 2. We shall refer to $S_c^{(0)}$ as a *level-0 pyramid with an apex c* . For any integer $p \geq 0$ define a *level- p pyramid* with an apex c as a cluster of 4 defects located at a set of cubes $S_c^{(p)} = \{c, c+2^p\hat{x}, c+2^p\hat{y}, c+2^p\hat{z}\}$. We note that a level- $(p+1)$ pyramid with an apex c can be represented as a sum (modulo two) of four level- p pyramids with an apex at $c, c+2^p\hat{x}, c+2^p\hat{y}$, and $c+2^p\hat{z}$. Therefore, $S_c^{(p)}$ can be created from the vacuum by an error $E_u^{(p)}$ defined recursively as

$$E_u^{(p+1)} = E_u^{(p)} E_{u+2^p\hat{x}}^{(p)} E_{u+2^p\hat{y}}^{(p)} E_{u+2^p\hat{z}}^{(p)}$$

with $E_u^{(0)} = XI_u$. Simple induction shows that $E_u^{(p)}$ acts on 4^p qubits. Its support can be regarded as a fractal object with a fractal dimension 2.

Suppose that the lattice has periodic boundary conditions and its linear size is $L = 2^n$ for some integer n . Then the four defects of a level- n pyramid cancel each other, i.e., $S_c^{(n)} = \emptyset$. It shows that the error $E_u^{(n)}$ is either a stabilizer or a logical operator. Consider an auxiliary operator $\bar{Z}_u = \prod_{i,j=1}^L ZI_{u-\hat{x}+i\hat{y}+j\hat{z}}$. One can easily check that \bar{Z}_u commutes with all stabilizer generators of X -type, see Fig. 2. On the other hand, \bar{Z}_u anti-commutes with $E_u^{(n)}$ since their supports overlap only on a single site $u + (2^n - 1)\hat{x} = u - \hat{x}$ and their restrictions on this site (ZI and XI respectively) anticommute. It shows that $E_u^{(n)}$ is a logical operator. Let us now show that $E_u^{(n)}$ has energy barrier at most $4 + 4 \log L$.

We claim that the error $E_u^{(p)}$ creating a level- p pyramid $S_c^{(p)}$ has energy barrier $\omega_p = 4p + 4$. Below we shall only consider error paths implementing single-qubit factors of $E_u^{(p)}$ in a certain order. We shall use induction to prove the following statement: *$E_u^{(p)}$ can be implemented by a*

sequence of single-qubit errors without creating more than ω_p defects. The apex c contains a defect after each error in this sequence. For $p = 0$, the claim is obvious. Suppose we have already proved the claim for $p = 0, \dots, q$. Let us build a level- $(q+1)$ pyramid $S_c^{(q+1)}$. We start from building a level- q pyramid $S_c^{(q)}$, which requires at most $4q+4$ defects and the apex c contains a defect after each elementary error. Now we can build another level- q pyramid $S_{c+2^q\hat{x}}^{(q)}$ using at most $4q+4$ defects plus the defects that were already present in $S_c^{(q)}$. Although $S_c^{(q)}$ has 4 defects, one of them is located at the apex of $S_{c+2^q\hat{x}}^{(q)}$. The induction hypothesis implies that any partial implementation of $S_{c+2^q\hat{x}}^{(q)}$ also contains a defect at $c+2^q\hat{x}$. These two defects cancel each other and must be subtracted from the total cost. Thus we can build two nearby pyramids at cost of $4 + (4q+4) - 2 = 4q+6$ defects. Both pyramids together have 6 defects. The same argument shows that building the third pyramid $S_{c+2^q\hat{y}}^{(q)}$ requires at most $6 + (4q+4) - 2 = 4q+8$ defects, and the three pyramids together have 6 defects. Building the last pyramid $S_{c+2^q\hat{z}}^{(q)}$ requires at most $6 + (4q+4) - 2 = 4q+8$ defects. Thus, we have constructed a pyramid $S_c^{(q+1)}$ using at most $4(q+1) + 4$ defects. Note that the apex c was occupied by a defect over all stages of this construction. It proves the step of induction. We conclude that a logical operator $E_u^{(n)}$ has energy barrier at most $4n+4 = 4+4\log L$.

Separation of charged clusters. We have shown that in a process of isolating a charged cluster, there is a logarithmic energy barrier. The following theorem quantifies how long the process must be.

Theorem 3. *Let E be a Pauli operator creating S , a neutral cluster of defects containing a charged cluster $S' \subseteq S$ of diameter r such that there are no other defects within distance R from S' . If $r+2R < L_{tqo}$, then the weight of E must be $\geq cR^\gamma$ for some constant $\gamma > 1$ and c .*

The proof again makes use of renormalization group, and shows that there is a subset of ‘fractal dimension’ $\gamma > 1$ in the support of E . We assume that E has weight minimum possible. Let w be an odd positive number. We say a set of sites $C \subseteq \Lambda$ is a *level- p chunk* if $\text{diam}(C) < w^p$. A *path* in the lattice is a finite sequence of sites (u_1, u_2, \dots, u_n) such that $d(u_i, u_{i+1}) = 1$. (Recall that we use the l_∞ metric d .) Equipped with paths, we can say whether a set is connected.

Definition 2. *A connected level- p chunk $C \subseteq S$ is maximal with respect to a set of sites S if there exist a connected subset $C^\circ \subseteq C$ and a path $\zeta = (u_1, \dots, m, \dots, u_n) \subseteq C^\circ$ satisfying*

- (i) $d(u_1, u_n) = w^p - w^{p-1}$,
- (ii) $d(u_1, m), d(u_n, m) \geq \frac{w^p - w^{p-1}}{2}$,

(iii) C° contains the connected component of m in $\mathcal{B}_{\frac{w^p - w^{p-1}}{2}}(m) \cap S$, and

(iv) C contains the connected component of C° in $\mathcal{B}_{\frac{w^p - 1}{2}}(C^\circ) \cap S$.

The last two conditions restricts the position of ζ in C such that ζ lies sufficiently far from the boundary of C . The site m will be referred to as a *midpoint* of C . Let S be the support of the Pauli operator E , any restriction of which obeys no-strings rule.

Lemma 3. *Given a path ζ in S joining u_1 and u_n such that $d(u_1, u_n) = lw^p - 1$, there are l disjoint maximal chunks of level- p whose midpoints are on ζ .*

Lemma 4. *For sufficiently large w , a maximal level- $(p+1)$ chunk C with respect to S admits a decomposition into $w+1$ or more maximal chunks of level- p with respect to S .*

The proofs of Lemma 3, 4 can be found in Appendix B. The support of the minimal Pauli operator E in Theorem 3 must admit a path connecting S' and $S \setminus S'$. Otherwise, S' can be regarded as being created locally, and our topological order condition demands the cluster be neutral. Since the path has length $\geq R$, Lemma 3 says we have a maximal chunk of level- p where p is such that $w^p \leq R < w^{p+1}$. Lemma 4 implies any maximal chunk of level- p must contain at least $(w+1)^p$ sites. This proves Theorem 3 with $\gamma = \frac{\log(w+1)}{\log w} > 1$. A similar argument proves the lower bound $d = \Omega(L^\gamma)$ on the code distance d of Code 1 in [27] since the minimal logical operator must contain a path of length L .

CONCLUSIONS AND OPEN PROBLEMS

We proposed an argument in favor of self-correcting properties for a class of 3D spin Hamiltonians with topological quantum order. Thermal diffusion of defects in these Hamiltonians is suppressed by the presence of logarithmic energy barriers. A novel technique for proving lower bounds on the energy barriers is presented.

Our results rise several questions. Firstly, Theorem 2 implies that the energy landscape contains a macroscopic number of local minimums separated by macroscopic energy barriers. These minimums correspond to low-lying excited states with only a few defects such that separation between defects is of order L . Such energy landscape suggests a possibility of a spin glass phase at sufficiently low temperature. We note that a spin glass phase can indeed be realized for some classical spin Hamiltonians with logarithmic energy barriers such as the model discovered by Newman and Moore [34]. Interplay between topological order and spin glassiness has been studied recently by several authors [24, 35]. Secondly, our paper

leaves open the question of how to perform error correction necessary to extract encoded information from a memory. In particular, no efficient error correction algorithm is known for the stabilizer code discovered in [27]. One could speculate however that locality and a macroscopic distance of the code are sufficient conditions for having a non-zero error threshold under random independent errors. Finally, an exciting open question is whether any 3D stabilizer code Hamiltonian with TQO has point-like topological defects. A proof of this conjecture would give an evidence that “strong self-correction” similar to the one in the 4D toric code is impossible for realistic physical systems.

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APPENDIX A

In this section we prove Lemma 2. For convenience of the reader, we repeat the statement of the lemma.

Lemma 2. *Let $S' \equiv S(t')$ and $S'' \equiv S(t'')$ be a consecutive pair of syndromes in the level- p syndrome history. Let E be the product of all errors E_j that occurred between S' and S'' . If $4m(2 + \xi(p)) < L_{tqo}$, then there exists an error \tilde{E} supported on $\mathcal{B}_{\xi(p)}(S' \cup S'')$ such that $E\tilde{E}$ is a stabilizer.*

Proof. Let us use induction in p . The base of induction is $p = 0$. In this case $E = E_j$ is a single-qubit error. If the qubit acted on by E does not belong to $\mathcal{B}_1(S' \cup S'')$, one must have $S' = S''$. It means that E is a single-qubit error with a trivial syndrome. Topological order condition implies that E is a stabilizer. Choosing $\tilde{E} = I$ proves the lemma for $p = 0$.

Suppose we have already proved the statement for some level p and let us prove for level $p+1$. Let $S' = S(t')$ and $S'' = S(t'')$ be consecutive syndromes in the level- $(p+1)$ history. Consider first the trivial case when $S' = S(t')$ and $S'' = S(t'')$ are also consecutive syndromes in the level- p history. Then S' and S'' are connected by a single level- p error E which, by induction hypothesis, has support on $\mathcal{B}_{\xi(p)}(S' \cup S'')$ (modulo stabi-

lizers). The latter is contained in $\mathcal{B}_{\xi(p+1)}(S' \cup S'')$ which proves the induction step.

The non-trivial case is when there is at least one level- p syndrome between S' and S'' . The interval of the level- p syndrome history between S' and S'' can be represented (after properly redefining the time variable t) as

$$S' \xrightarrow{E_{lead}} S(1) \xrightarrow{E_1} S(2) \xrightarrow{E_2} \dots \xrightarrow{E_{\tau-1}} S(\tau) \xrightarrow{E_{tail}} S''.$$

Here all syndromes $S(1), \dots, S(\tau)$ are sparse at the level p and all transitions are caused by single level- p errors. The condition of sparsity implies that the set of elementary cubes occupied by $S(t)$ has a unique partition into a disjoint union of clusters $C_a(t)$ such that each cluster has diameter at most $\xi(p)$ and the distance between any pair of clusters (if any) is at least

$$\begin{aligned} \text{dist}(C_a(t), C_b(t)) &\geq \xi(p+1) - 2\xi(p) \geq (10\alpha - 2)\xi(p) \\ &\geq 8\alpha\xi(p). \end{aligned}$$

Represent any intermediate syndrome as a disjoint union

$$S(t) = S^c(t) \cup S^n(t), \quad t = 1, \dots, \tau, \quad (3)$$

where $S^c(t)$ and $S^n(t)$ include all charged and all neutral clusters $C_a(t)$ respectively. Let g be the number of clusters in $S^c(t)$. We claim that g does not depend on t . Indeed, since a level- p error E_t acts on $\xi(p)$ -neighborhood of $S(t) \cup S(t+1)$, the sparsity condition implies that E_t cannot create/annihilate a charged cluster $C_a(t)$ from the vacuum, or map a charged cluster to a neutral cluster (or vice versa). The same argument shows that each cluster $C_a(t) \subseteq S^c(t)$ can ‘move’ at most by $\xi(p)$ per time step, that is, we can parameterize

$$S^c(t) = C_1(t) \cup \dots \cup C_g(t)$$

such that a ‘world-line’ of the a -th charged cluster obeys the continuity condition

$$\text{dist}(C_a(t+1), C_a(t)) \leq \xi(p). \quad (4)$$

We can now use the no-strings rule to show that all charged clusters are ‘locked’ near their initial positions, so that their world-lines are essentially parallel to the time axis. More precisely, we claim that

$$\text{dist}(C_a(t), C_a(1)) \leq \alpha\xi(p) \quad \text{for all } 1 \leq t \leq \tau. \quad (5)$$

Indeed, suppose Eq. (5) is false for some a . Using the continuity Eq. (4) one can find a time step t_1 such that $\text{dist}(C_a(t_1), C_a(1)) > \alpha\xi(p)$ and $\text{dist}(C_a(t), C_a(1)) \leq \alpha\xi(p)$ for all $1 \leq t < t_1$. Let E_{close} be the product of all level- p errors E_j that occurred between $S(1)$ and $S(t_1)$ within distance $(2 + \alpha)\xi(p)$ from $C_a(1)$. Since all intermediate syndromes are sparse at level p , the net effect of E_{close} is to annihilate the charged cluster $C_a(1)$ and create the charged cluster $C_a(t_1)$. Equivalently, applying E_{close} to the vacuum creates a pair of charged

clusters $C_a(1)$ and $C_a(t_1)$. However, this contradicts to the no-strings rule since $C_a(1)$ and $C_a(t_1)$ have linear size at most $\xi(p)$ while the distance between them is greater than $\alpha\xi(p)$. Thus we have proved Eq. (5).

We will say that $\vec{x} \in \Lambda$ is *close to S'* iff $\vec{x} \in \mathcal{B}_{\xi(p+1)}(S')$. We will say that $\vec{x} \in \Lambda$ is *close to S''* iff $\vec{x} \in \mathcal{B}_{\xi(p+1)}(S'')$.

Let E_t be the level- p error causing the transition from $S(t)$ to $S(t+1)$, where $t = 1, \dots, \tau-1$. Let E_t^c is the restriction of E_t onto $\mathcal{B}_{\xi(p)}(S^c(t) \cup S^c(t+1))$, and E_t^n is the restriction of E_t onto $\mathcal{B}_{\xi(p)}(S^n(t) \cup S^n(t+1))$. The sparsity of $S(t)$ and localization of level- p errors then implies that

$$E_t = E_t^c \cdot E_t^n. \quad (6)$$

We claim that *any error E_t^c is close to S'* . Indeed, each cluster in $S^c(1)$ is within distance $2\xi(p)$ from S' since otherwise a single level- p error E_{lead} would be able to create a charged cluster from the vacuum. Using Eq. (5) we infer that $C_a(t) \in \mathcal{B}_{(2+\alpha)\xi(p)}(S')$ for all $a = 1, \dots, g$. Therefore E_t^c is close to S' .

We shall now define a 'localized' leading error \tilde{E}_{lead} that maps the syndrome S' to $S^c(1)$ such that the support of \tilde{E}_{lead} is close to S' . For any neutral cluster $C \in S^n(1)$ let $O'(C)$ be a Pauli operator creating C from the vacuum. By definition of a neutral cluster, we can assume that $O'(C)$ has support in $\mathcal{B}_1(C)$. Set

$$\tilde{E}_{lead} = E_{lead} \cdot G \cdot \prod_{C \in S^n(1)} O'(C),$$

where $G \in \mathcal{G}$ is a stabilizer to be chosen later. Since E_{lead} is a single level- p error, we have two options:

- (a) any cluster $C \in S^n(1)$ is within distance $\xi(p)$ from S' ,
- (b) there is exactly one cluster $C_{far} \in S^n(1)$ such that the distance between C_{far} and S' is greater than $\xi(p)$.

In case (a) we set $G = I$ if the support of E_{lead} is close to S' and $G = E_{lead}$ otherwise (if E_{lead} is not close to S' , but any cluster $C \in S^n(1)$ is within distance $\xi(p)$ from S' , the error E_{lead} creates no defects when applied to the vacuum, that is, E_{lead} must be a stabilizer). In case (b) the error E_{lead} creates a neutral cluster C_{far} from the vacuum. In this case we will set $G = E_{lead}O'(C_{far})$. We conclude that the support of \tilde{E}_{lead} is close to S' and \tilde{E}_{lead} maps S' to $S^c(1)$.

We can apply the same rules to the error E_{tail} and the syndrome $S^n(\tau)$. It allows us to find a stabilizer G and define a localized error

$$\tilde{E}_{tail} = E_{tail} \cdot G \cdot \prod_{C \in S^n(\tau)} O''(C),$$

such that \tilde{E}_{tail} maps $S^c(\tau)$ to S'' and the support of \tilde{E}_{tail} is close to S'' . The operator $O''(C)$ above creates a neutral cluster $C \in S^n(\tau)$ from the vacuum.

We can now define a localized level- $(p+1)$ error \tilde{E} whose support is close to $S' \cup S''$ as

$$\tilde{E} = \tilde{E}_{lead} \cdot E_1^c \cdots E_{\tau-1}^c \cdot \tilde{E}_{tail}. \quad (7)$$

By construction, it describes an error path

$$S' \xrightarrow{\tilde{E}_{lead}} S^c(1) \xrightarrow{E_1^c} S^c(2) \xrightarrow{E_2^c} \cdots \xrightarrow{E_{\tau-1}^c} S^c(\tau) \xrightarrow{\tilde{E}_{tail}} S''.$$

It remains to check that $E \cdot \tilde{E}$ is a stabilizer. Combining Eq. (6) and Eq. (7) we conclude that

$$E \cdot \tilde{E} = (\tilde{E}_{lead} E_{lead}) \cdot E_1^n \cdots E_{\tau-1}^n \cdot (\tilde{E}_{tail} E_{tail}).$$

Applying $E \cdot \tilde{E}$ to the vacuum generates the following chain of transitions:

$$\begin{aligned} \text{vac} &\xrightarrow{\tilde{E}_{lead} E_{lead}} S^n(1) \xrightarrow{E_1^n} S^n(2) \xrightarrow{E_2^n} \cdots \\ &\cdots \xrightarrow{E_{\tau-1}^n} S^n(\tau) \xrightarrow{\tilde{E}_{tail} E_{tail}} \text{vac}. \end{aligned} \quad (8)$$

Here all transitions are caused by errors whose support can be enclosed by at most m cubes of linear size $2 + \xi(p)$. Each syndrome $S^n(t)$ consists of at most m neutral clusters of diameter $\xi(p)$, i.e., it can be created from the vacuum by an error whose support can be enclosed by at most m cubes of linear size $2 + \xi(p)$. Now the statement that $E \cdot \tilde{E}$ is a stabilizer follows from the following proposition.

Proposition 1. *Let Q_j be Pauli operators causing a chain of transitions*

$$\text{vac} \xrightarrow{Q_1} S_1 \xrightarrow{Q_2} S_2 \xrightarrow{Q_3} \cdots \xrightarrow{Q_r} S_r \xrightarrow{Q_{r+1}} \text{vac}.$$

Let P_j be some Pauli operator creating the syndrome S_j from the vacuum. Suppose the support of any operator P_j and any operator Q_j can be enclosed by at most n cubes of linear size R such that $4nR < L_{tqo}$. Then the product $\bar{Q} \equiv Q_1 \cdots Q_r Q_{r+1}$ is a stabilizer.

Proof of Proposition 1. Let ψ_0 be any ground state. Define a sequence of states

$$\begin{aligned} \psi(1) &= P_1 Q_1 \cdot \psi_0, \\ \psi(j+1) &= (P_j P_{j+1}) Q_{j+1} \cdot \psi(j), \\ \psi(r+1) &= Q_{r+1} P_r \cdot \psi(r), \end{aligned}$$

for $j = 0, \dots, r-1$. Obviously,

$$\psi(j) = P_j \cdot (Q_1 \cdots Q_j) \cdot \psi_0 \quad \text{for } j = 1, \dots, r$$

and $\psi(r+1) = \bar{Q} \cdot \psi_0$. It follows that all states $\psi(j)$ are ground states, and the transition from $\psi(j)$ to $\psi(j+1)$ can be caused by a Pauli operator

$$O_j = P_j P_{j+1} Q_{j+1}.$$

Let M_j be the support of O_j . Obviously, M_j can be enclosed by at most $3n$ cubes of linear size R . Suppose

first that M_j is a connected set, i.e., one can connect any pair of qubits from M_j by a path (u_1, \dots, u_l) such that the distance between u_a and u_{a+1} is 1. Then M_j can be enclosed by a single cube of linear size at most $3nR$. Our assumptions on n and R , and the topological order condition then implies that O_j is a stabilizer. Suppose now that M_j consists of several disconnected components M_j^α , such that the distance between any pair of components is at least 2. Let O_j^α be the restriction of O_j onto a connected component M_j^α . Locality of the stabilizer generators G_a and the fact that O_j commutes with any G_a implies that O_j^α commutes with any G_a . Furthermore, M_j^α can be enclosed by a cube of linear size at most $3nR$. Topological order condition then implies that O_j^α is a stabilizer. It follows that all operator O_j are stabilizers, that is, $\psi(j+1) = \psi(j)$ for all j . However, it means that $\bar{Q}\psi_0 = \psi_0$ for any ground state ψ_0 , that is, \bar{Q} is a stabilizer. \square

APPENDIX B

In this section we prove the technical lemmas needed for Theorem 3.

Lemma 3. *Given a path ζ in S joining u_1 and u_n such that $d(u_1, u_n) = lw^p - 1$, there are l disjoint maximal chunks of level- p whose midpoints are on ζ .*

Proof. For convenience, we assume that the z -coordinates of u_1 and u_n are 0 and $lw^p - 1$, respectively. Consider $l+1$ planes P_i perpendicular to the z -axis, whose z -coordinates are iw^p for $i = 0, 1, \dots, l$. In each region between the two consecutive planes P_{i-1} and P_i , there is a subpath $\zeta_i = (u_{j_{i-1}}, \dots, u_{j_i})$ such that $d(u_{j_{i-1}}, u_{j_i}) = w^p - 1$. Choose $m_i \in \zeta_i$ such that $d(u_{j_{i-1}}, m_i), d(m_i, u_{j_i}) \geq \frac{w^p-1}{2}$. Let C_i be the connected component of m_i within $S \cap \mathcal{B}_{\frac{w^p}{2}}(m_i)$. Add, if necessary, some points of ζ_i to C_i to get a maximally connected C'_i . This C'_i is a maximal chunk of sites with midpoint being m_i . Any two C'_i 's are disjoint since each of them lies in a unique region enclosed by P_{i-1} and P_i . \square

Lemma 4. *For sufficiently large w , a maximal level- $(p+1)$ chunk C with respect to S admits a decomposition into $w+1$ or more maximal chunks of level- p with respect to S .*

Proof. Recall that S is the support of the Pauli operator E , any restriction of which obeys no-strings rule. Define the *boundary* of a subset U of S to be $\partial U = \mathcal{B}_1(U) \cap U^c \cap S$. Then, any subset U of sites with boundary enclosed in a two disjoint regions can be regarded as a string segment.

By the definition of the maximal chunk, there exists a path $(u_1, \dots, m, \dots, u_n)$ in $C^\circ \subseteq C$ such that

$d(u_1, u_n) = w^{p+1} - w^p$. We assume that the z -coordinates of u_1, u_n differ by $w^{p+1} - w^p$. We will show that there are sufficiently long and separated paths in C° , to which we apply Lemma 3 to find $w+1$ maximal chunks of level- p . They will lie in $\mathcal{B}_{\frac{w^p}{2}}(C^\circ)$, and hence in C .

Let $M(N)$ be the subset of S consisted of sites whose z -coordinates differ from that of $u_1(u_n)$ by at most ηw^p . First, suppose ∂C° is not contained in $M \cup N$. Since $u_1 \in M$ and $u_n \in N$, there is a site $s \in C^\circ$ adjacent (of distance 1) to ∂C° such that $d(s, u_1), d(s, u_n) > \eta w^p$. Furthermore, $d(s, m) \geq \frac{w^{p+1}-w^p}{2} - 1$; Otherwise, C° contains a site in the boundary, which is a contradiction.

Consider the shortest network \mathcal{N} of paths in C° connecting four sites u_1, m, u_n, s . (The length of a network of paths is the number of sites in the union of the paths.) Let ζ be the shortest path in \mathcal{N} from u_1 to u_n . If s is not contained in $\mathcal{B}_{3w^p}(\zeta)$, then $\zeta' \subseteq \mathcal{N}$ joining s to a site on ζ has a subpath $\zeta'' \subseteq \zeta'$ of diameter at least $2w^p$ such that ζ'' is separated from ζ by w^p . Applying Lemma 3 to ζ and ζ'' , we find at least $w+1$ maximal chunk of level- p . If m is not contained in $\mathcal{B}_{3w^p}(\zeta)$, a similar argument reveals at least $w+1$ maximal chunk of level- p .

Suppose both s and m are contained in $\mathcal{B}_{3w^p}(\zeta)$. Observe that $\zeta \setminus (\mathcal{B}_{4w^p}(s) \cup \mathcal{B}_{4w^p}(m))$ consists of three connected components $\zeta_1, \zeta_2, \zeta_3$, two of which have diameter $\geq \frac{w^{p+1}-w^p}{2} - 8w^p$ and the other has diameter $\geq (\eta-4)w^p$. Two distinct $\mathcal{B}_{\frac{w^p}{2}}(\zeta_i)$ and $\mathcal{B}_{\frac{w^p}{2}}(\zeta_j)$ ($i, j = 1, 2, 3$) do not overlap because of the minimality of ζ . Applying Lemma 3, we find $w + \eta - 21$ maximal chunks of level- p . Choosing $\eta > 21$, we get the desired result.

Next, suppose ∂C° is contained in $M \cup N$. Let $s_M, s_N \in C^\circ \setminus (M \cup N)$ be sites adjacent to M and N , respectively. The separation between M and N is $(w-1-2\eta)w^p$. If it is greater than $\eta'\alpha w^p$, there must be a z -plane P that contains s_M or s_N such that $P \cap C^\circ$ has diameter $> \eta'w^p$; Otherwise, the no-strings rule is violated. Let $v_1, v_2 \in P \cap C^\circ$ be sites separated by $\eta'w^p$. The four sites, u_1, u_n, v_1, v_2 are sufficiently separated and connected by some paths in C° . Arguing as before, we find $(w-1)+\eta'-16$ maximal chunks of level- p . The choice of $\eta' > 17$ and $w > 1 + 2\eta + \eta'\alpha$ proves the lemma. \square

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